III Economic Growth (Continued)

D. Endogenous Growth: Romer's (1986) Model

1 Basic Idea

- A variant of AK model: Using learning-by-doing (investing) to eliminate the tendency for diminishing returns by assuming that knowledge creation is a side product of investment.

2 Firms

- The labour-augmenting neoclassical production function for firm i

  $$Y_i = F(K_i, A_i L_i)$$

  where $L = \Sigma_i L_i$ is constant. Assume that productivity A_i growth comes from learning-by-doing that works through each firm's investment and that each firm's knowledge is a *public good* that other firms can access at zero cost.

- The assumptions about productivity growth implies that A_i can be written as $A_i = K$ and

  $$Y_i = F(K_i, KL_i)$$

  where each firm faces diminishing returns to capital K_i: $\partial F / \partial K_i > 0$; $\partial^2 F / \partial K_i^2 < 0$.

- Let $f(k_i, K) \equiv F(K_i/L_i, K)$ and $y_i \equiv Y_i/L_i = f(k_i, K)$, then a firm's profit is

  $$L_i[f(k_i, K) - (r + \delta)k_i - w].$$

Profit maximization gives

$$f_1(k_i, K) = \partial y_i/\partial k_i = r + \delta$$

$$f(k_i, K) - k_i f_1(k_i, K) = \partial Y_i/\partial L_i = w.$$

- In equilibrium, all firms choose the same level of capital, so that we have $k_i = k$ and $K = kL$. Since $f(k_i, K)$ is homogeneous of degree one in (k_i, K), we have

$$f(k_i, K)/k_i = \tilde{f}(K/k_i) = \tilde{f}(L),$$

where $f(k_i, K)/k_i$ or $\tilde{f}(L)$ is the average product of capital (APK). $\tilde{f}(L)$ satisfies $\tilde{f}'(L) > 0$ and $\tilde{f}''(L) < 0$. Note L is constant and so is APK $\tilde{f}(L)$ because the learning-by-doing eliminates decreasing returns to capital.

- Differentiating $f(k_i, K)/k_i = \tilde{f}(K/k_i)$ with respect to k_i, we get

$$f_1(k_i, K)/k_i - f(k_i, K)/k_i^2 = \tilde{f}'(K/k_i)(-K/k_i^2),$$

which implies

$$f_1(k_i, K) = \tilde{f}(L) - L\tilde{f}'(L)$$

or equivalently,

$$r = f_1(k_1, K) - \delta = \tilde{f}(L) - L\tilde{f}'(L) - \delta.$$

So the marginal product of capital (MPK)f_1 is below APK and MPK is increasing in L (because $\tilde{f}''(L) < 0$).

3 Households

The households' problem is given by

$$\max_c \int_0^\infty e^{-\rho t}\left(\frac{c^{1-\theta}-1}{1-\theta}\right)dt$$

subject to

$$\dot{a} = ra + w - c$$

and

$$\lim_{t\to\infty} a e^{-\int_0^t r(v)dv} \geq 0.$$

The optimal condition is

$$\gamma_c = \dot{c}/c = (r-\rho)/\theta$$

and the transversality condition is

$$\lim_{t\to\infty} a e^{-\int_0^t r(v)dv} = 0.$$

4 Equilibirum

- Equilibrium conditions: (i) $a = k$ (hence $\dot{a} = \dot{k}$); (ii). $k_i = k$.
- From Sections 2 and 3, we have

  $$\gamma_c = \dot{c}/c = (1/\theta)[\tilde{f}(L) - L\tilde{f}'(L) - \delta - \rho]$$

  where γ_c is constant as long as L is constant and γ_c is increasing in L (the *scale effect*).
- To obtain $\gamma_k = \dot{k}/k$, we rewrite the household budget constraint

  $$\dot{k} = rk + w - c.$$

Since

$$w = f(k, K) - kf_1(k, K) = \tilde{f}(L)k - k[\tilde{f}(L) - L\tilde{f}'(L)]$$

and

$$r = \tilde{f}(L) - L\tilde{f}'(L) - \delta$$

, we have

$$\dot{k} = \tilde{f}(L)k - c - \delta k$$

or

$$\gamma_k = \dot{k}/k = \tilde{f}(L) - \delta - c/k.$$

Note that $\dot{a}/a = \dot{k}/k = \dot{c}/c$ and there is no transitional dynamics.

- The restrictions on parameter values for finite utility:

  $$\int_0^\infty e^{-\rho t}(c^{1-\theta} - 1)/(1-\theta)dt.$$

  Solving $\dot{c}/c = (r - \rho)/\theta$ gives

  $$c = c(0)e^{(1/\theta)(r-\rho)t}.$$

  Substituting this into the utility function yields

  $$\int_0^\infty e^{-\rho t}\left(\frac{c^{1-\theta} - 1}{1-\theta}\right) dt = \int_0^\infty e^{(1/\theta)[-\rho+(1-\theta)r]t}\left(\frac{c(0)^{1-\theta}}{1-\theta}\right) dt$$

  $$-\int_0^\infty e^{-\rho t}\frac{1}{1-\theta}dt.$$

  This is bounded if and only if (iff)

  $$\rho > (1-\theta)r$$

  or equivalently,

  $$\rho > (1-\theta)[\tilde{f}(L) - L\tilde{f}'(L) - \delta].$$

5 Pareto Nonoptimality and Policy Implications

- Reason for non-optimality: The externality in the form of the learning-by-doing is taken as given when an individual producer makes decisions and hence the result is not Pareto optimal.
- The planner internalizes the spillovers of knowledge across firms. The Planner's Problem is

  $$\max_c \int_0^\infty e^{-\rho t}(c^{1-\theta}-1)/(1-\theta)dt$$

  subject to $\dot{k} = \tilde{f}(L)k - c - \delta k$ and $\lim_{t\to\infty} e^{-rt}k(t) \geq 0$.
  The Hamiltonian is

  $$H = e^{-\rho t}(c^{1-\theta}-1)/(1-\theta) + \lambda[\tilde{f}(L)k - c - \delta k].$$

  The necessary optimal conditions are:

  $$H_c = 0 \quad \Rightarrow \quad e^{-\rho t}c^{-\theta} = \lambda, \quad \Rightarrow \quad \lambda(\rho + \theta\dot{c}/c) = -\dot{\lambda},$$

  $$H_k = -\dot{\lambda} \quad \Rightarrow \quad \lambda[\tilde{f}(L) - \delta] = -\dot{\lambda},$$

  $$\lim_{t\to\infty}(\lambda k) = 0 \quad \text{(TVC)}.$$

  These equations imply:

  $$\gamma_c = \dot{c}/c = [\tilde{f}(L) - \delta - \rho]/\theta.$$

- Comparing the social planner's solution with the decentralized equilibrium, we have

  $$\gamma_{c,\text{planner}} > \gamma_{c,\text{decentralized}},$$

  because

  $$\gamma_{c,\text{planner}} - \gamma_{c,\text{decentralized}} = L\tilde{f}'(L)/\theta > 0.$$

This is because, as mentioned earlier, the planner internalizes the spillovers of knowledge and invests more than decentralized firms who ignore the spillovers.

6 A Cobb-Douglas Example

- Assume that $Y_i = AK_i^{\alpha}(KL_i)^{1-\alpha}, \quad 0 < \alpha < 1$. Then $y_i = Y_i/L_i = Ak_i^{\alpha}K^{1-\alpha}$; $k_i = K_i/L_i$; and $k = K/L$.
- In equilibrium, $k_i = k$,

$$y/k = f(k,K)/k = \tilde{f}(L) = AL^{1-\alpha}$$

$$\frac{\partial Y_i}{\partial K_i} = f_1(k,K) = \tilde{f}(L) - L\tilde{f}'(L) = A\alpha L^{1-\alpha}(\text{constant MPK})$$

$$\gamma_{c,\text{decentralized}} = (A\alpha L^{1-\alpha} - \delta - \rho)/\theta$$

$$\gamma_{c,\text{planner}} = (AL^{1-\alpha} - \delta - \rho)/\theta > \gamma_{c,\text{decentralized}}$$

since $\alpha < 1$.

- Government interventions to correct the nonoptimality of the decentralized economy: subsidizing capital (investment-tax credits) or subsidizing production with a lump-sum tax or consumption tax (without labor-leisure choice).

  Suppose the government gives investment-tax credit on capital renting by lump-sum taxes T (s per unit of capital):

$$\text{Profits} = L_i[f(k_i,K) - (r+\delta-s)k_i - w].$$

  Firms' optimal condition: $\partial\text{Profits}/\partial k_i = 0$ leads to $f_1(k_i,K) = r+\delta-s$ and $w = f(k_i,K) - (r+\delta-s)k_i$.

Correspondingly, $f_1(k_i, k) = \tilde{f}(L) - L\tilde{f}'(L)$; $f(k_i, K) = k_i\tilde{f}(L)$. Then $r = \tilde{f}(L) - L\tilde{f}'(L) - \delta + s$ and $w = k_i\tilde{f}(L) - (r+\delta-s)k_i$. In equilibrium $k_i = k$; $sK = Lsk = LT$ (or $sk = T$); $a = k$.

- If $s = L\tilde{f}'(L)$, then the decentralized economy will obtain Pareto optimality (i.e., identical to the social planner's problem). We need to show that (i) $\gamma_{c,\text{decentralized}} = \gamma_{c,\text{planner}}$ (optimality) and (ii) the resource constraint for the decentralized equilibrium is the same as that for the social planner(feasibility).

  (i). $\gamma_{c,\text{decentralized}} = (r-\rho)/\theta = [\tilde{f}(L) - L\tilde{f}'(L) - \delta + s - \rho]/\theta = [\tilde{f}(L) - \delta - \rho]/\theta = \gamma_{c,\text{planner}}$ if $s = L\tilde{f}'(L)$ (Note in the Cobb-Douglas case, $s/\tilde{f}(L) = 1 - \alpha$.)

  (ii). In the decentralized economy with $s = L\tilde{f}'(L)$ and $sk = T$,

$$\dot{k} = rk + w - c - T = rk + k\tilde{f}(L) - (r + \delta - s)k - c - T$$

$$= k\tilde{f}(L) - c - \delta k,$$

  which is the same as that facing the social planner.

7 Contributions and Problems

- Endogenous growth over time is achieved through learning-by-doing accumulation of knowledge.
- The model exhibits "scale effects": the growth rate of per capita income rises as population grows, which may not be supported by evidence. It also abstracts from other sources of growth.