III Economic Growth (Continued)

C The Ramsey-Cass-Koopmans Model

1 Literature

- Ramsey (1928)
- Cass (1965) and Koopmans (1965)

2 Households (Preferences)

- Population growth: $L(0) = 1$, $L(t) = e^{nt}$ ($n > 0$ is the growth rate of population)
- Preferences:

  $$\int_0^\infty u(c(t))e^{nt}e^{-\rho t}dt$$

  where ρ is the rate of time preferences ($\rho > n$) and $u(c(t))$ satisfies $u' > 0$, $u'' < 0$, $\lim_{c\to 0} u' = \infty$ and $\lim_{c\to\infty} u' = 0$.
- Intertemporal budget constraint:

  $$\dot{a} = w + ra - c - na$$

  where a is net assets held by households, w is wage income per worker and r is the interest rate.
- Non-Ponzi-game condition:

  $$\lim_{t\to\infty} ae^{-\int_0^t (r(v)-n)dv} \geq 0$$

  That is, the household's debt per person (the negative value of a) cannot grow as fast as or faster than $r - n$.

3 The Household's Problem

$$\max_c \int_0^\infty u(c)e^{nt}e^{-\rho t}dt$$

subject to

$$\dot{a} = w + ra - c - na$$

$$\lim_{t\to\infty} ae^{-\int_0^t (r(v)-n)dv} \geq 0$$

- The Hamiltonian:

$$H = e^{-(\rho-n)t}u(c) + \lambda(w + ra - c - na)$$

- The optimal conditions:

$$H_c = 0: \quad e^{(n-\rho)t}u'(c) = \lambda \quad (*)$$

$$H_a = -\dot{\lambda}: \quad (r-n)\lambda = -\dot{\lambda}$$

$$\dot{a} = w + ra - c - na$$

$$\lim_{t\to\infty} a\lambda = 0$$

- The Euler Equation: Differentiating (*) with respect to t gives

$$(n-\rho)e^{(n-p)t}u'(c) + e^{(n-\rho)t}u''(c)\dot{c} = \dot{\lambda}$$

$$= (n-r)\lambda = (n-r)e^{(n-\rho)t}u'(c),$$

which leads to

$$r = \rho - u''\dot{c}/u' = \rho - (u''c/u')\dot{c}/c$$

where the term $-(u''c/u')$ is the elasticity of marginal utility or the coefficient of relative risk aversion or the inverse of elasticity of intertemporal substitution and the term $-(u''\dot{c}/u') = -(du'/dt)/u'$ is the rate of change in marginal utility of consumption u'.

- Interpretation of the Euler Equation: Choose consumption so as to equate the rate of return to savings r to the rate of time preference ρ plus the rate of decrease of the marginal utility of consumption u' (due to growing consumption)

- Note: if $r = \rho$, then $\dot{c}/c = 0$; if $r > \rho$, then $\dot{c}/c > 0$; if $r < \rho$, then $\dot{c}/c < 0$

- Example: Assume $u(c_t) = \frac{c_t^{1-\theta}-1}{1-\theta}$ with $\theta > 0$ (a CIES utility function), where θ is the magnitude of the elasticity of marginal utility and $\sigma = 1/\theta$ is the (constant) elasticity of intertemporal substitution.

  ‡ The Euler equation

  $$\dot{c}/c = (r - p)/\theta$$

  ‡ A higher $\sigma = 1/\theta$ means stronger willingness to substitute intertemporally, or larger responsiveness of $\dot{c}/c$ to the gap between r and ρ.

- Transversality condition: Solving $(r - n)\lambda = -\dot{\lambda}$ gives $\lambda = \lambda(0)e^{-t(r-n)} > 0$, where $\lambda(0) = u'(c(0)) > 0$. So we have

  $$\lim_{t\to\infty} \lambda = \lim_{t\to\infty} \lambda(0)e^{-\int_0^t (r(v)-n)dv} = 0,$$

  which leads to the transversality condition

  $$\lim_{t\to\infty} ae^{-\int_0^t (r(v)-n)dv} = 0. \quad \text{(TVC)}$$

This condition implies that a does not grow asymptotically at a rate as high as $r-n$, or equivalently, that the level of assets does not grow as at a rate as high as r.

- The consumption function: Define the average interest rate between 0 and t as

$$\bar{r}(t) = (1/t)\int_0^t r(v)dv,$$

then solving the flow budget constraint, $\dot{a} = w + ra - c - na$ gives the intertemporal budget constraint (for any $T \geq 0$):

$$a(T)e^{-[\bar{r}(t)-n]T} + \int_0^T c(t)e^{-[\bar{r}(t)-n]t}dt = a(0) + \int_0^T w(t)e^{-[\bar{r}(t)-n]t}dt.$$

As $T \to \infty$, $a(T)e^{-[\bar{r}(t)-n]T} \to 0$ (the transversality condition). Then the intertemporal budget constraint becomes

$$\int_0^\infty c(t)e^{-[\bar{r}(t)-n]t}dt = a(0) + \int_0^\infty w(t)e^{-[\bar{r}(t)-n]t}dt.$$

This condition says that the present value of the consumption stream equals the sum of the initial assets and the present values of the wage income stream.

Solving $\dot{c}/c = (r-\rho)/\theta$ gives

$$c = c(0)e^{[(\bar{r}(t)-\rho)/\theta]t}.$$

Substituting this into the intertemporal budget constraint for c leads to the consumption function

$$c = \mu(0)e^{[(\bar{r}(t)-\rho)/\theta]t}[a(0) + \int_0^\infty w(t)e^{-(\bar{r}(t)-n)t}dt]$$

where $\mu(0) = 1/\int_0^\infty e^{[r(1-\theta)/\theta-\rho/\theta+n]t}dt$ is propensity to consume out of wealth. Note the income and substitution effects of $\bar{r}(t)$.

4 Firms

- No technological progress $x = 0$
- Neoclassical technology: $Y = F(K, L)$. Denote $y = Y/L$, and $k = K/L$. Then, we have $y = f(k)$
- Let R be the rental price for a unit of capital services, then the net rate of return to capital owners is $R - \delta$.
- Capital and loans are perfect substitutes, so $r = R - \delta$
- Profit $= F(K, L) - (r + \delta)K - wL = L[f(k) - (r + \delta)k - w]$
- Maximising profits by choosing k gives

  $$f'(k) = r + \delta$$

- Perfect competition leads to zero profits, implying

  $$w = [f(k) - kf'(k)]$$

5 Competitive Equilibrium

- Firms and households take r and w as given.
- In closed economy: $a = k$, so $\dot{k} = w + rk - c - nk$. Then substituting the expressions for w and r into the above equation gives

  $$\dot{k} = f(k) - c - (n + \delta)k \qquad (A)$$

- Substituting $r = f'(k) + \delta$ into the Euler equation leads

  $$\dot{c}/c = (1/\theta)[f'(k) - \delta - \rho] \quad (B)$$

6 The Steady State

- The steay-state growth rates of k and c are zero

  ‡ Using $a = k$ and the interest rate equation, we have $\lim_{t\to\infty} ke^{-t[f'(k)-\delta-n]} = 0$ implying

  $$f'(k) > \delta + n.$$

  Denote $\gamma_k^* = \dot{k}/k$ as the steady-state growth rate. Then $k = k(0)\exp(\gamma_k^*)$. Consequently, the transversality condition becomes

  $$\lim_{t\to\infty} k(0)\exp\left[-t(f'(k) - \delta - n - \gamma_k^*)\right] = 0$$

  which holds only if, $f'(k) > \delta + n + \gamma_k^*$.

  ‡ From (A), we have

  $$c = f(k) - (n+\delta)k - k\gamma_k^* \qquad (A')$$

  . Differentiating (A') with respect to t leads to

  $$\dot{c} = \dot{k}[f'(k) - (n + \delta + \gamma_k^*)],$$

  which implies that $\dot{c}$ and $\dot{k}$ must have the same sign.

  ‡ $\gamma_k^* = \gamma_c^* = 0$. This is because: (i) If $\gamma_k^* > 0$, then eventually $k \to \infty$ and $f'(k) \to 0$, leading to $f'(k) - \delta - \rho < 0$. As a result, $\gamma_c^* = \dot{c}/c = [f'(k) - \delta - \rho]/\theta < 0$, contradicting sign $\gamma_c^* =$ sign γ_k^*. (ii) If $\gamma_k^* < 0$, then eventually $f'(k) \to \infty$. Consequently, $\gamma_c = \dot{c}/c = [f'(k) - \delta - \rho]/\theta > 0$. This contradicts sign $\gamma_c^* =$ sign γ_k^*.

  ‡ In the steady-state equilibrium, k, c, and y are constant.

- The key reason the neoclassical growth model does not have long-run endogenous growth is diminishing marginal product, i.e. $f'(k) \to 0$ if $k \to \infty$, or $f''(k) < 0$. When capital per worker rises, its marginal contribution to output and the rate of return to saving fall to zero, eliminating incentives for saving and investment.

- Denote the steady state as (k^*, c^*, y^*) with $\dot{c} = 0$ and $\dot{k} = 0$. The solution is given by:

$$f'(k^*) = \delta + \rho \text{ (the modified golden rule)}$$

$$c^* = f(k^*) - (\delta + n)k^*$$

$$y^* = f(k^*)$$

$$r^* = f'(k^*) - \delta$$

$k^* < k_{\text{gold}}$. This is because: (i) $f'(k_{\text{gold}}) = n + \delta$ from $\max_k c$ subject to $\dot{k} = 0$. (ii) According to the transversality condition, $f'(k^*) > n + \delta$. (iii) By (i) and (ii), $f'(k^*) > f'(k_{\text{gold}})$. (iv) Since $f''(\cdot) < 0$, (iii) implies $k^* < k_{\text{gold}}$. The implication is that inefficient oversaving cannot occur in the Cass-Koopman framework, while undersaving prevails in this model compared to the golden rule.

7 Transitional Dynamics

- The differential equations (A) and (B) (along with the traverality condition (TVC)) determine the transitional dynamics of the model.

– Three procedures for solving a system of differential equations: $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$.

  ‡ Analytical solution

  ‡ Phase Diagram

  ‡ Numerical solution

– Phase diagram: The system has to be an autonomous system, that is, the variable t does not enter into the f and g functions as a separate argument. (Figure a)

  ‡ The Phase Space: two-dimensional space

  ‡ The Demarcation Curves: $\dot{x} = 0$ and $\dot{y} = 0$.

$$f(x, y) = 0 \quad [\dot{x} = 0]$$

$$g(x, y) = 0 \quad [\dot{y} = 0]$$

‡ The Slopes of the Demarcation Curves: The slopes of the $\dot{x} = 0$ and $\dot{y} = 0$ curves are respectively

$$\frac{dy}{dx}|_{\dot{x}=0} = -\frac{f_x}{f_y}, \quad (f_y \neq 0)$$

$$\frac{dy}{dx}|_{\dot{y}=0} = -\frac{g_x}{g_y}, \quad (g_y \neq 0)$$

‡ Four Regions of the Phase Space: The two demarcation curves divide the phase space into four distinct regions. The intersection point is the intertemporal equilibrium of the system (both x and y are stationary). At any other point, either x or y (or both) would be changing over time, in directions dictated by the signs of the time derivatives $\dot{x}$ and

$\dot{y}$ at that point. Assuming that $f_x < 0, f_y > 0, g_x > 0$ and $g_y < 0$, we have

$$\frac{\partial \dot{x}}{\partial x} = f_x < 0$$

$$\frac{\partial \dot{y}}{\partial y} = g_y < 0.$$

These derivatives generate the plus or minus sign on both sides of the demarcation curves.

‡ Directional Arrows: intertemporal movement of x and y.

‡ Streamlines (Phase Trajectories or Phase Paths): dynamic movement of the system from any initial point.

‡ Types of Equilibria: (i) Nodes; (ii) Saddle Points; (iii) Forci or Focuses; (iv) Vortices or Vortexes.

(i) A node is an equilibrium such that all the streamlines associated with it either flow noncyclically toward it (stable node) or flow noncyclically away from it (unstable node). (Figure b)

(ii) A saddle point is an equilibrium with exactly one pair of streamlines (the stable branches of the saddle point) flowing directly and consistently toward the equilibrium and exactly one pair of streamlines (the unstable branches) flowing directly and consistently away from it (all other trajectories heading toward the saddle point initially but sooner or later turning away from it). (Figure c)

(iii) A focus is an equilibrium characterized by whirling trajectories, all of which either flow cyclically toward it (stable focus) or flow cyclically away from it (unstable focus). (Figure d)

(iv) A vortex (or center) is an equilibrium with whirling streamlines forming a family of loops (concentric circles or ovals) orbiting around the equilibrium in a perpetual motion. (Figure e)

‡ Back to the Ramsey-Cass-Koopmans Model: Figure f.

– Numerical Solution: Time-Elimination and Projection Methods.

‡ Considering the following system of differential equation: $\dot{k} = k^{0.3} - c$ and $\dot{c} = c[0.3k^{-0.7}] - 0.006$ with given $k(0)$ and $\lim_{t\to}[e^{-0.006t}k] = 0$.

‡ Using the time-elimination method to get a differential equation for the policy function, $c = c(k)$:

$$c'(k) = \dot{c}/\dot{k} = \frac{c(k)[0.3k^{-0.7} - 0.06]}{k^{0.3} - c(k)}.$$

‡ Solving this differential equation to obtain the policy function $c = c(k)$:

(i) Using Chebyshev polynomials to approximate c with a linear functional:

$$c(k; a) = \sum_{i=1}^{n} a_i \psi_i(k)$$

where n is the number of terms used and ψ_i is given by

$$\psi_i(k) = T_{i-1}(x(k)), \quad x(k) \equiv \left(\frac{2(k - k_{min})}{k_{max} - k_{min}} - 1\right)$$

where $k \in [k_{min}, k_{max}]$ and $T_i(x)$ is a Chebyshev polynomial, i.e. $T_i(x) = cos(i \cdot arccos(x))$. $T_i(x)$ satisfies the following:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x).$$

(ii) Choosing a such that $R(k;a) = c'(k) - \frac{c(k)[0.3k^{-0.7}-0.06]}{k^{0.3}-c(k)}$ is zero at a finite number of points.

(iii) Selecting n values of k, i.e. $(k_1, k_2, ..., k_n)$, as the n zeros of T_n. This gives n equations: $R(k_i;a) = 0$, $i = 1..n$. Solving these equations gives $(a_1, a_2, ..., a_n)$, we then have $c = c(k)$.

(iv) Substituting $c = c(k)$ into $\dot{k} = k^{0.3} - c$ and solving this equation for k. (Other variables such y are functions of k).

(v) Example:

$$n = 9, \quad k_{ss} = 9.96617657819346$$

$$k_{max} = 5k_{ss}/3, \; k_{min} = k_{ss}/3$$

$$(k_1, k_2, ..., k_9) =$$

$$(3.42299793678749, 4.21220184802705,$$

$$5.69542003113658, 7.69375448387792,$$

$$9.96617657861491, 12.2385986698854,$$

$$14.2369331322315, 15.7201512989131,$$

$$16.5093552242669)$$

$$(a_1, a_2, ..., a_9) =$$

$$(1.94191206451309, 0.538471336115668,$$

$$-0.0539349304884249, 0.0111837376711720,$$

$$-0.00282635923132117, 0.000787920584893088,$$

$$-0.000237421813650450, 0.0000829168326194185,$$

$$-0.0000227419451339622).$$

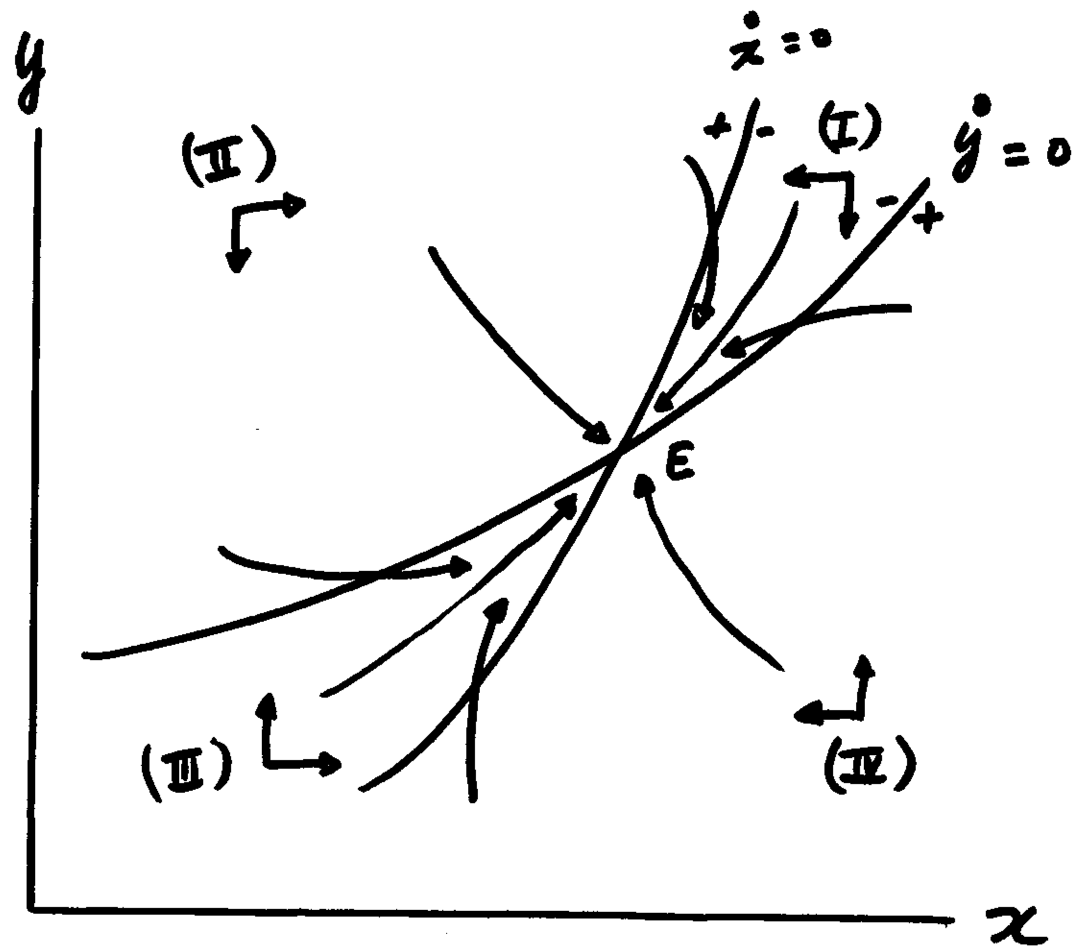

Fig. a

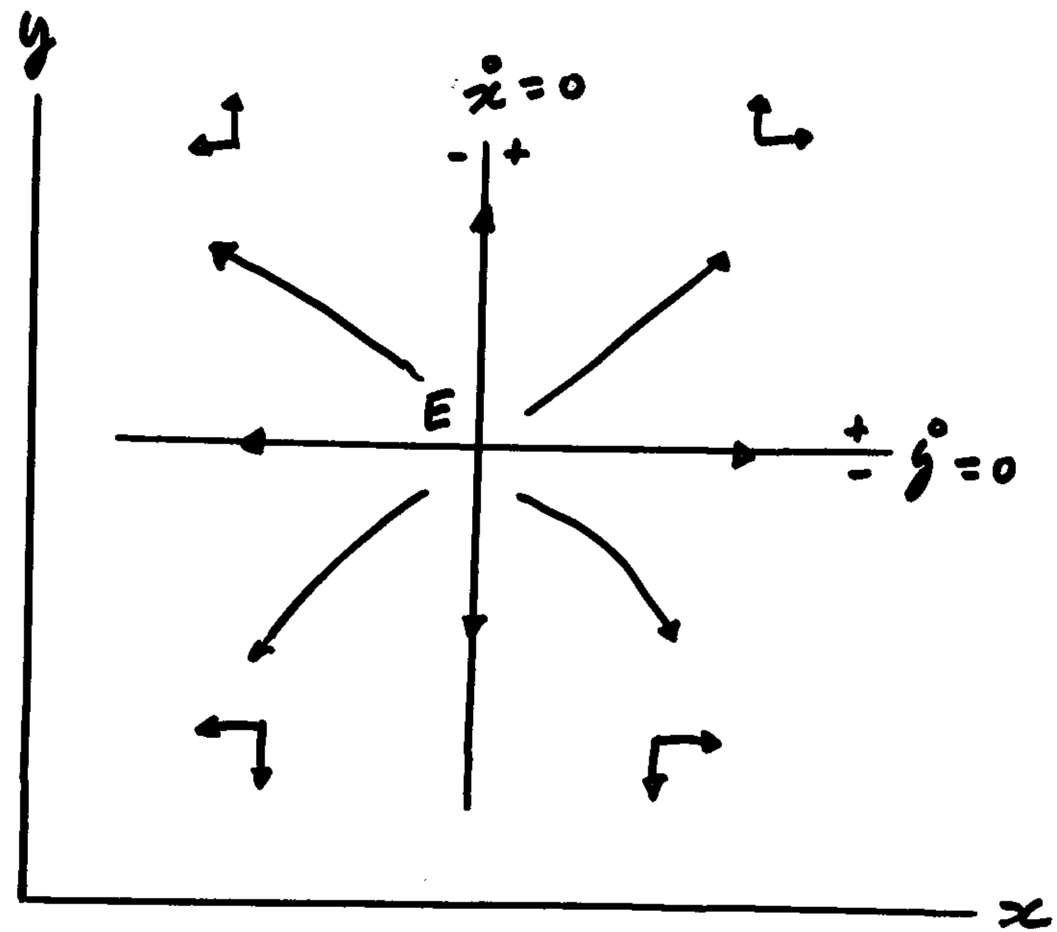

Fig. b

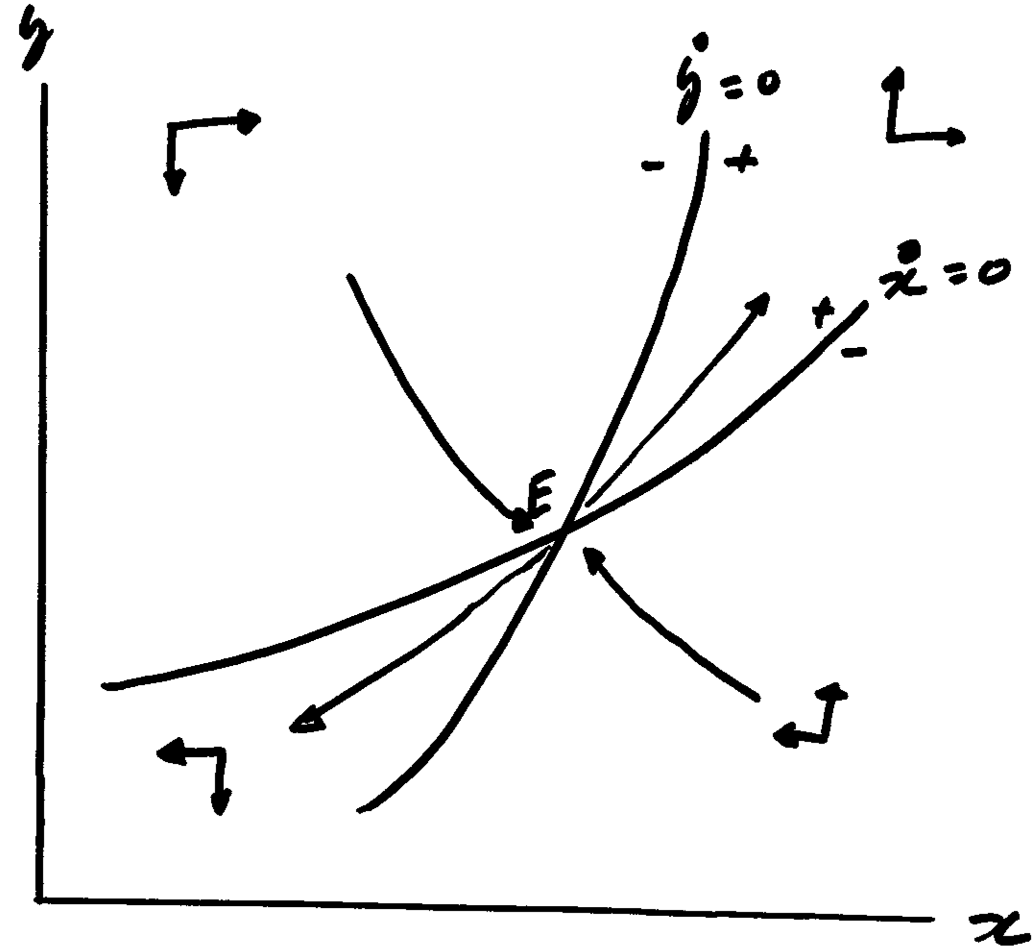

Fig. c

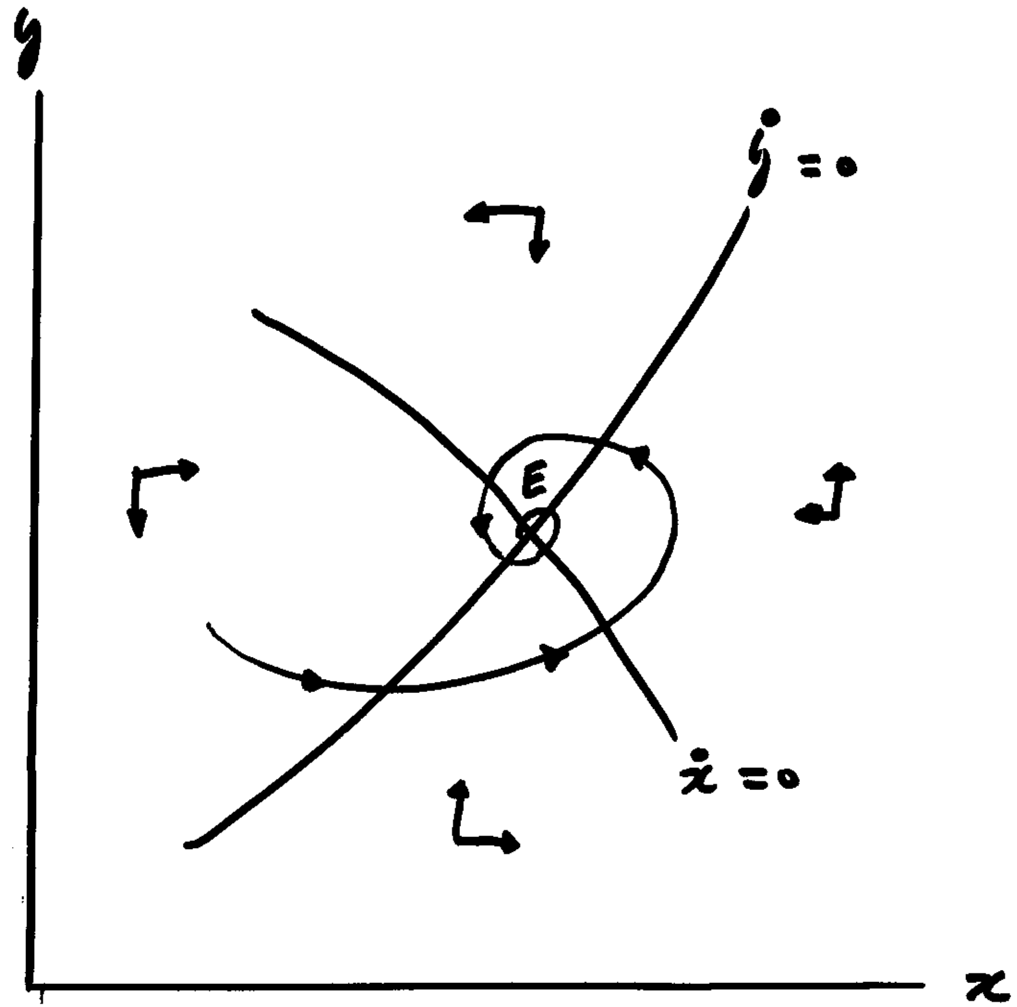

Fig. d

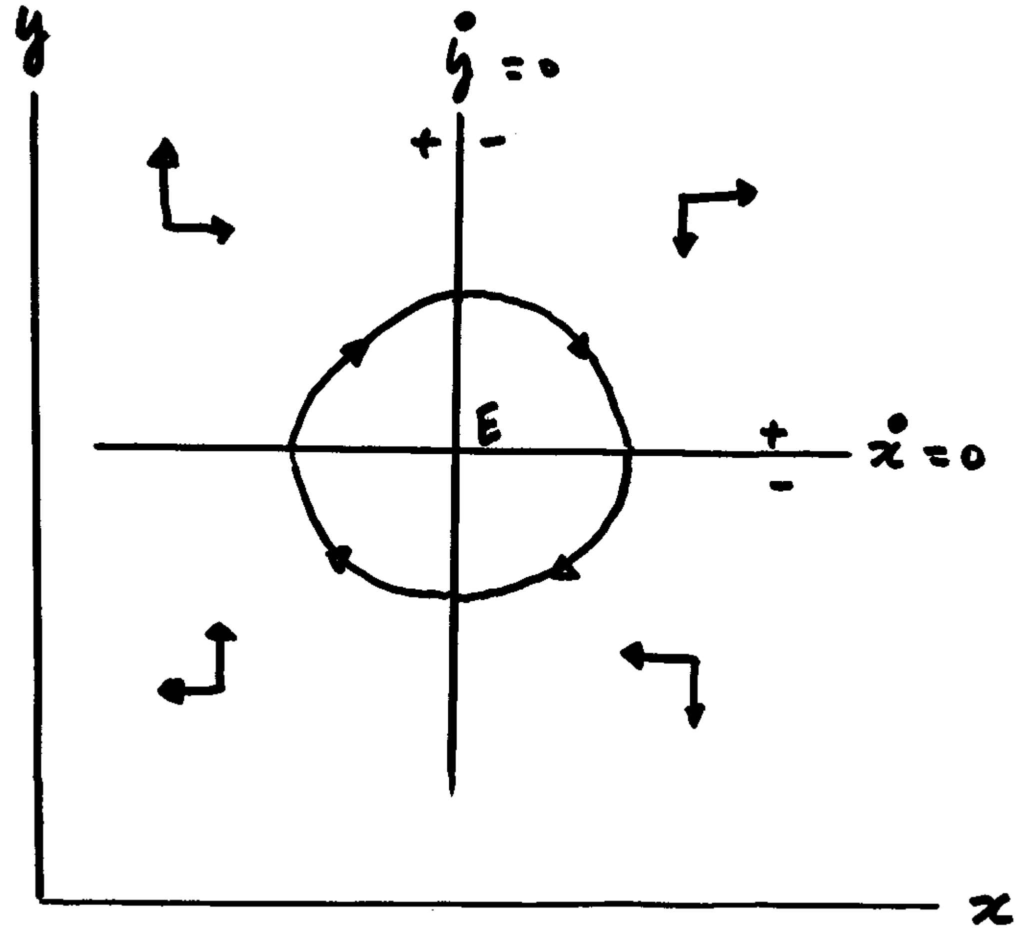

Fig. e

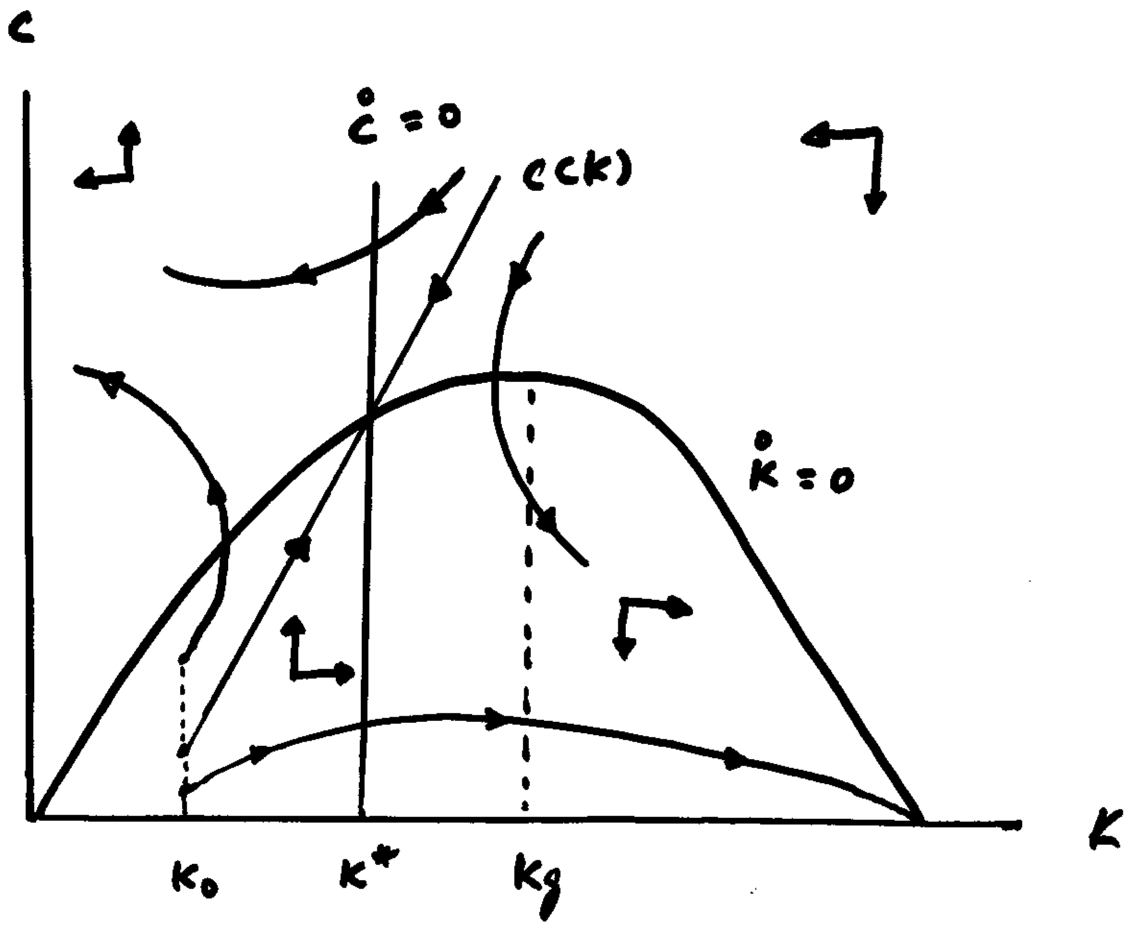

Fig. f

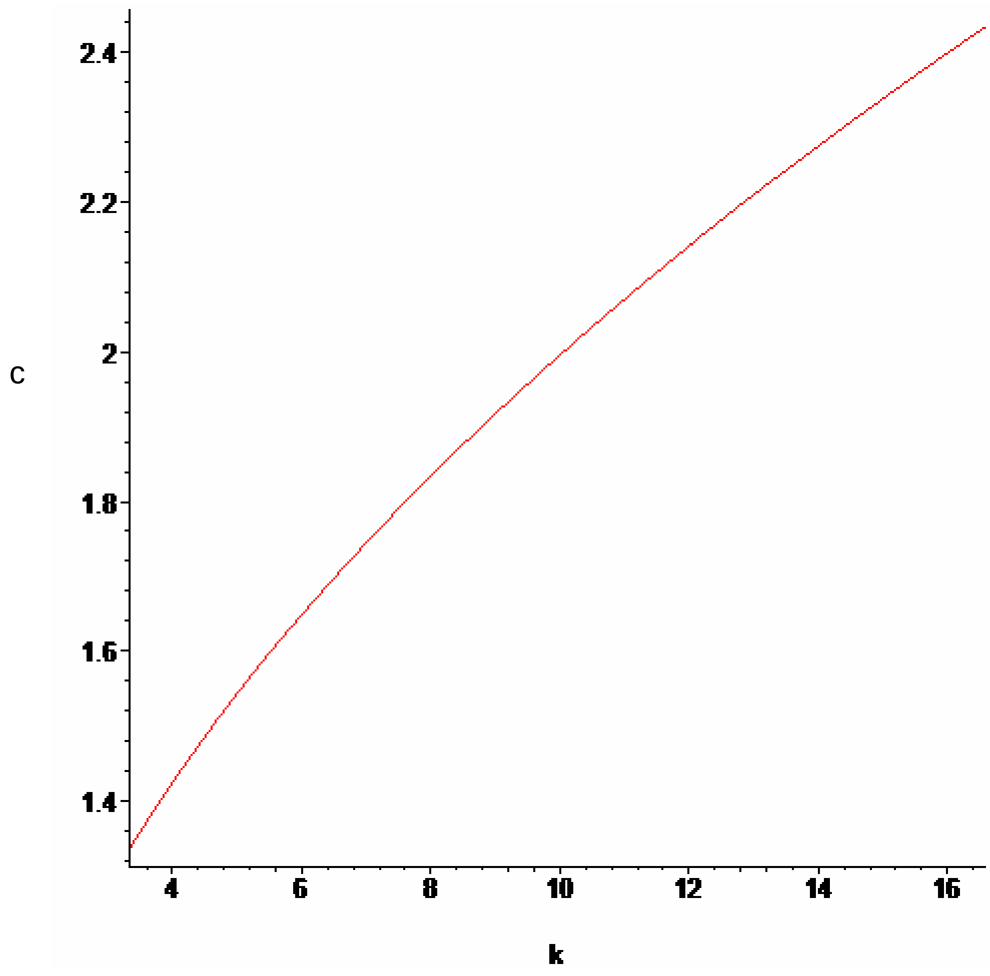

Figure g: Policy Function c=c(k)

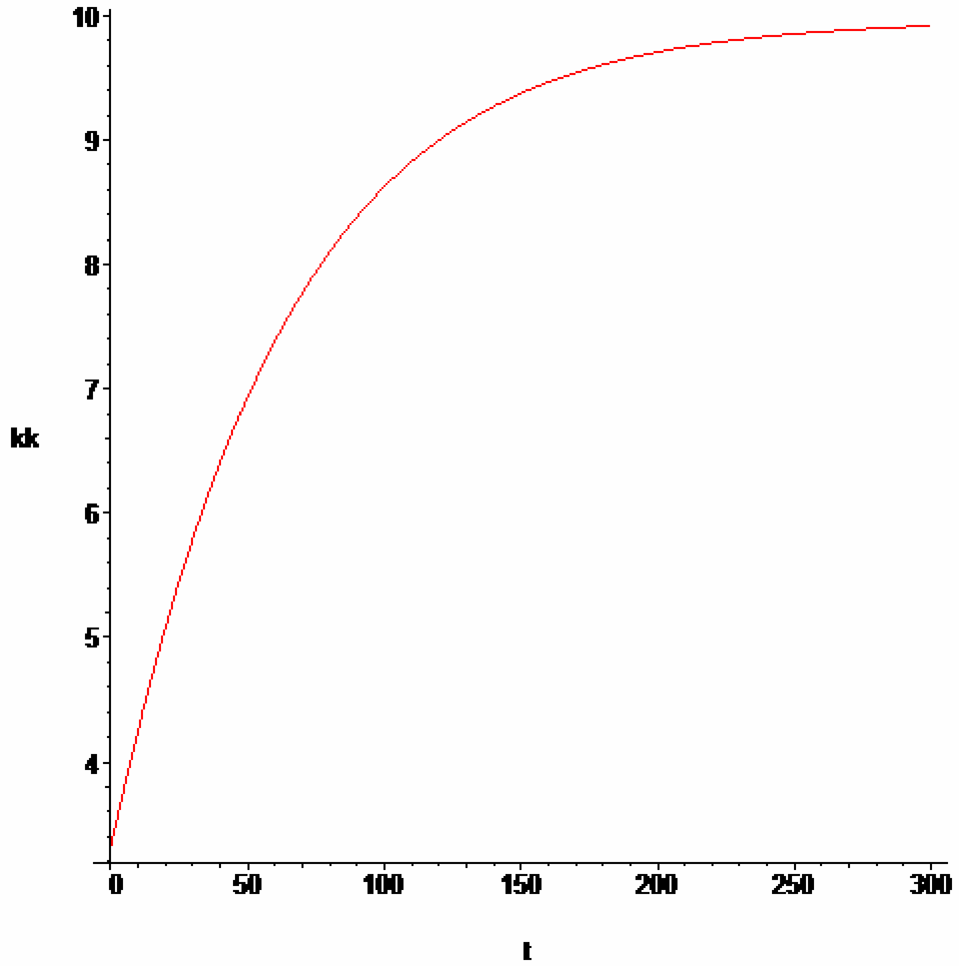

Figure h: Time Path of Capital k